

**THE SET OF COMMON FIXED POINTS OF A
ONE-PARAMETER CONTINUOUS SEMIGROUP OF MAPPINGS
IS $F(T(1)) \cap F(T(\sqrt{2}))$**

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ABSTRACT. In this paper, we prove the following theorem: Let $\{T(t) : t \geq 0\}$ be a one-parameter continuous semigroup of mappings on a subset C of a Banach space E . The set of fixed points of $T(t)$ is denoted by $F(T(t))$ for each $t \geq 0$. Then

$$\bigcap_{t \geq 0} F(T(t)) = F(T(1)) \cap F(T(\sqrt{2}))$$

holds. Using this theorem, we discuss convergence theorems to a common fixed point of $\{T(t) : t \geq 0\}$.

1. INTRODUCTION

Let C be a subset of a Banach space E , and let T be a nonexpansive mapping on C , i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We know that T has a fixed point in the case that E is uniformly convex and C is bounded, closed and convex; see Browder [5], Göhde [9], and Kirk [13]. We denote by $F(T)$ the set of fixed points of T .

Let τ be a Hausdorff topology on E . A family of mappings $\{T(t) : t \geq 0\}$ is called a one-parameter τ -continuous semigroup of mappings on C if the following are satisfied:

- (sg 1) $T(s+t) = T(s) \circ T(t)$ for all $s, t \geq 0$;
- (sg 2) for each $x \in X$, the mapping $t \mapsto T(t)x$ from $[0, \infty)$ into C is continuous with respect to τ .

As topology τ , we usually consider the strong topology of E . Also, a family of mappings $\{T(t) : t \geq 0\}$ is called a one-parameter τ -continuous semigroup of nonexpansive mappings on C (in short, nonexpansive semigroup) if (sg 1), (sg 2) and the following (sg 3) are satisfied:

- (sg 3) for each $t \geq 0$, $T(t)$ is a nonexpansive mapping on C .

We know that nonexpansive semigroup $\{T(t) : t \geq 0\}$ has a common fixed point in the case that E is uniformly convex and C is bounded, closed and convex; see Browder [5]. Moreover, in 1974, Bruck [8] proved that nonexpansive semigroup $\{T(t) : t \geq 0\}$ has a common fixed point in the case that C is weakly compact, convex, and has the fixed point property for nonexpansive mappings.

In this paper, we prove the following theorem: Let $\{T(t) : t \geq 0\}$ be a one-parameter τ -continuous semigroup of mappings on a subset C of a Banach space

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E for some Hausdorff topology τ on E . Then

$$\bigcap_{t \geq 0} F(T(t)) = F(T(1)) \cap F(T(\sqrt{2}))$$

holds. Using this theorem, we discuss convergence theorems to a common fixed point of nonexpansive semigroups $\{T(t) : t \geq 0\}$.

2. PRELIMINARIES

Throughout this paper we denote by \mathbb{Q} the set of rational numbers, and by \mathbb{N} the set of positive integers. For real number t , we denote by $[t]$ the maximum integer not exceeding t . It is obvious that for each real number t , there exists $\varepsilon \in [0, 1)$ such that $t = [t] + \varepsilon$.

We recall that a Banach space E is called strictly convex if $\|x + y\|/2 < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A Banach space E is called uniformly convex if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x + y\|/2 < 1 - \delta$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$. It is clear that a uniformly convex Banach space is strictly convex. The norm of E is called Fréchet differentiable if for each $x \in E$ with $\|x\| = 1$, $\lim_{t \rightarrow 0} (\|x + ty\| - \|x\|)/t$ exists and is attained uniformly in $y \in E$ with $\|y\| = 1$.

The following Lemma is the corollary of Bruck's result in [7].

Lemma 1 (Bruck [7]). *Let C be a subset of a strictly convex Banach space E . Let S and T be nonexpansive mappings from C into E with common fixed point. Then for each $\lambda \in (0, 1)$, a mapping U from C into E defined by $Ux = \lambda Sx + (1 - \lambda)Tx$ for $x \in C$ is nonexpansive and $F(U) = F(S) \cap F(T)$ holds.*

Proof. It is obvious that $F(U) \supset F(S) \cap F(T)$. Fix $x \in F(U)$ and $w \in F(S) \cap F(T)$. Then we have

$$\begin{aligned} \|x - w\| &= \|\lambda Sx + (1 - \lambda)Tx - w\| \\ &\leq \lambda \|Sx - w\| + (1 - \lambda) \|Tx - w\| \\ &\leq \lambda \|x - w\| + (1 - \lambda) \|x - w\| \\ &= \|x - w\| \end{aligned}$$

and hence

$$\|x - w\| = \|\lambda Sx + (1 - \lambda)Tx - w\| = \|Sx - w\| = \|Tx - w\|.$$

So, from the strict convexity of E , we obtain

$$\lambda Sx + (1 - \lambda)Tx = Sx = Tx.$$

Hence $x \in F(S) \cap F(T)$. This completes the proof. \square

The following four convergence theorems for nonexpansive mappings are well-known.

Theorem 1 (Baillon [2]). *Let C be a bounded closed convex subset of a Hilbert space E . Let T be a nonexpansive mapping on C . Let $x \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_n = \frac{Tx + T^2x + T^3x + \cdots + T^n x}{n}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to a fixed point of T .

Theorem 2 (Reich [17]). *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable. Let T be a nonexpansive mapping on a bounded closed convex subset C of E . Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and $x_{n+1} = \alpha_n T x_n + (1 - \alpha_n) x_n$ for $n \in \mathbb{N}$. where $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. Then $\{x_n\}$ converges weakly to a fixed point of T .*

Theorem 3 (Browder [6]). *Let C be a bounded closed convex subset of a Hilbert space E , and let T be a nonexpansive mapping on C . Let $\{\lambda_n\}$ be a sequence in $(0, 1)$ converging to 0. Fix $u \in C$ and define a sequence $\{x_n\}$ in C by $x_n = (1 - \lambda_n) T x_n + \lambda_n u$ for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a fixed point of T .*

Theorem 4 (Wittmann [24]). *Let C be a bounded closed convex subset of a Hilbert space E , and let T be a nonexpansive mapping on C . Let $u \in C$ and define a sequence $\{x_n\}$ in C by $x_1 \in C$ and $x_{n+1} = (1 - \lambda_n) T x_n + \lambda_n u$ for $n \in \mathbb{N}$, where $\{\lambda_n\}$ is a sequence in $[0, 1]$ satisfying the following:*

$$\lim_{n \rightarrow \infty} \lambda_n = 0; \quad \sum_{n=1}^{\infty} \lambda_n = \infty; \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$$

Then $\{x_n\}$ converges strongly to a fixed point of T .

3. LEMMAS

In this Section, we prove two lemmas, which are used in Section 4.

Lemma 2. *Let t be a nonnegative real number and let $\{\beta_n\}$ be a sequence in $(0, \infty)$ converging to 0. Define sequences $\{\delta_n\}$ in $[0, \infty)$ and $\{k_n\}$ in $\mathbb{N} \cup \{0\}$ as follows:*

- $\delta_1 = t$;
- $k_n = [\delta_n / \beta_n]$ for $n \in \mathbb{N}$;
- $\delta_{n+1} = \delta_n - k_n \beta_n$ for $n \in \mathbb{N}$.

Then the following hold:

- (i) $0 \leq \delta_{n+1} < \beta_n$ for all $n \in \mathbb{N}$;
- (ii) $k_n \in \mathbb{N} \cup \{0\}$ for all $n \in \mathbb{N}$;
- (iii) $\{\delta_n\}$ converges to 0;
- (iv) $\sum_{j=1}^n k_j \beta_j + \delta_{n+1} = t$ for all $n \in \mathbb{N}$; and
- (v) $\sum_{j=1}^{\infty} k_j \beta_j = t$.

Proof. We put $\varepsilon_n \in [0, 1)$ with

$$\frac{\delta_n}{\beta_n} = k_n + \varepsilon_n$$

for $n \in \mathbb{N}$. We have

$$\delta_{n+1} = \delta_n - k_n \beta_n = \varepsilon_n \beta_n < \beta_n$$

for all $n \in \mathbb{N}$. From this, we also have $\delta_{n+1} = \varepsilon_n \beta_n \geq 0$. This implies (i). It is obvious that (ii) and (iii) follow from (i). Let us prove (iv). We have

$$k_1 \beta_1 + \delta_2 = k_1 \beta_1 + (\delta_1 - k_1 \beta_1) = \delta_1 = t.$$

We assume (iv) holds for some $n \in \mathbb{N}$. Then we have

$$\begin{aligned} \sum_{j=1}^{n+1} k_j \beta_j + \delta_{n+2} &= \sum_{j=1}^{n+1} k_j \beta_j + (\delta_{n+1} - k_{n+1} \beta_{n+1}) \\ &= \sum_{j=1}^n k_j \beta_j + \delta_{n+1} \\ &= t. \end{aligned}$$

So, by induction, we have (iv). From (iii) and (iv), we have

$$\sum_{n=1}^{\infty} k_n \beta_n = t.$$

This completes the proof. \square

Lemma 3. *Let α and β be positive real numbers satisfying $\alpha/\beta \notin \mathbb{Q}$. Define sequences $\{\alpha_n\}$ in $(0, \infty)$ and $\{k_n\}$ in \mathbb{N} as follows:*

- $\alpha_1 = \max\{\alpha, \beta\};$
- $\alpha_2 = \min\{\alpha, \beta\};$
- $k_n = \lfloor \alpha_n / \alpha_{n+1} \rfloor$ for all $n \in \mathbb{N};$
- $\alpha_{n+2} = \alpha_n - k_n \alpha_{n+1}$ for all $n \in \mathbb{N}.$

Then the following hold:

- (i) $0 < \alpha_{n+1} < \alpha_n$ for all $n \in \mathbb{N};$
- (ii) $k_n \in \mathbb{N}$ for all $n \in \mathbb{N};$
- (iii) $\alpha_n / \alpha_{n+1} \notin \mathbb{Q}$ for all $n \in \mathbb{N};$ and
- (iv) $\{\alpha_n\}$ converges to 0.

Proof. We note that (i) implies (ii). By the assumption of $\alpha/\beta \notin \mathbb{Q}$, we have $\alpha \neq \beta$. Hence

$$\alpha_1 = \max\{\alpha, \beta\} > \min\{\alpha, \beta\} = \alpha_2 > 0.$$

It is obvious that $\alpha_1/\alpha_2 \notin \mathbb{Q}$. We assume that $0 < \alpha_{j+1} < \alpha_j$ and $\alpha_j/\alpha_{j+1} \notin \mathbb{Q}$ for some $j \in \mathbb{N}$. Since $\alpha_{j+2} = \alpha_j - k_j \alpha_{j+1}$, we have

$$\frac{\alpha_{j+2}}{\alpha_{j+1}} = \frac{\alpha_j}{\alpha_{j+1}} - k_j \notin \mathbb{Q}$$

and hence $\alpha_{j+1}/\alpha_{j+2} \notin \mathbb{Q}$. Put $\varepsilon_j \in [0, 1)$ satisfying

$$\frac{\alpha_j}{\alpha_{j+1}} = k_j + \varepsilon_j.$$

Since $\alpha_j/\alpha_{j+1} \notin \mathbb{Q}$, we note that $\varepsilon_j > 0$. We have

$$\alpha_{j+2} = \alpha_j - k_j \alpha_{j+1} = \varepsilon_j \alpha_{j+1} < \alpha_{j+1}.$$

From this, we also have $\alpha_{j+2} = \varepsilon_j \alpha_{j+1} > 0$. Therefore we have shown that $0 < \alpha_{j+2} < \alpha_{j+1}$ and $\alpha_{j+1}/\alpha_{j+2} \notin \mathbb{Q}$. By induction, we have (i), (ii) and (iii). Let us prove (iv). Since $\{\alpha_n\}$ is a sequence of positive real numbers and strictly decreasing, $\{\alpha_n\}$ converges to some $\alpha_\infty \in [0, \infty)$. We assume $\alpha_\infty > 0$. Then we can choose $j \in \mathbb{N}$ such that

$$\alpha_\infty < \alpha_{j+1} < \alpha_j < 2\alpha_\infty.$$

We have

$$k_j = \left\lfloor \frac{\alpha_j}{\alpha_{j+1}} \right\rfloor = 1 \quad \text{and} \quad \alpha_{j+2} = \alpha_j - k_j \alpha_{j+1} = \alpha_j - \alpha_{j+1} < \alpha_\infty.$$

This is a contradiction. Therefore $\alpha_\infty = 0$ and this implies (iv). This completes the proof. \square

4. MAIN RESULTS

In this Section, we give our main results. We know the following.

Proposition 1. *Let E be a Banach space and let τ be a Hausdorff topology on E . Let $\{T(t) : t \geq 0\}$ be a one-parameter τ -continuous semigroup of mappings on a subset C of E . Let $\{\alpha_n\}$ be a sequence in $[0, \infty)$ converging to $\alpha_\infty \in [0, \infty)$, and satisfying $\alpha_n \neq \alpha_\infty$ for all $n \in \mathbb{N}$. Suppose that $z \in C$ satisfies*

$$T(\alpha_n)z = z$$

for all $n \in \mathbb{N}$. Then z is a common fixed point of $\{T(t) : t \geq 0\}$.

Proof. We note that

$$T(\alpha_\infty)z = \tau\text{-}\lim_{n \rightarrow \infty} T(\alpha_n)z = z.$$

We put

$$\beta_n = |\alpha_n - \alpha_\infty| > 0$$

for $n \in \mathbb{N}$. By the assumption, $\{\beta_n\}$ is a sequence in $(0, \infty)$ converging to 0. Since

$$\max\{\alpha_n, \alpha_\infty\} = \min\{\alpha_n, \alpha_\infty\} + \beta_n,$$

we have

$$\begin{aligned} T(\beta_n)z &= T(\beta_n) \circ T(\min\{\alpha_n, \alpha_\infty\})z \\ &= T(\beta_n + \min\{\alpha_n, \alpha_\infty\})z = T(\max\{\alpha_n, \alpha_\infty\})z \\ &= z \end{aligned}$$

for all $n \in \mathbb{N}$. We also have

$$T(0)z = T(0) \circ T(\alpha_1)z = T(0 + \alpha_1)z = T(\alpha_1)z = z.$$

Fix $t > 0$. Then by Lemma 2, there exists a sequence $\{k_n\}$ in $\mathbb{N} \cup \{0\}$ such that

$$\sum_{n=1}^{\infty} k_n \beta_n = t.$$

For each $n \in \mathbb{N}$ with $\sum_{j=1}^n k_j \beta_j > 0$, we obtain

$$\begin{aligned} T\left(\sum_{j=1}^n k_j \beta_j\right)z &= T(\beta_n)^{k_n} \circ T(\beta_{n-1})^{k_{n-1}} \circ \dots \circ T(\beta_2)^{k_2} \circ T(\beta_1)^{k_1} z \\ &= T(\beta_n)^{k_n} \circ T(\beta_{n-1})^{k_{n-1}} \circ \dots \circ T(\beta_2)^{k_2} z \\ &= \dots = T(\beta_n)^{k_n} z \\ &= z, \end{aligned}$$

where $T(\beta_j)^0$ is the identity mapping on C . Hence, we have

$$T(t)z = \tau\text{-}\lim_{n \rightarrow \infty} T\left(\sum_{j=1}^n k_j \beta_j\right) z = z.$$

This completes the proof. \square

We now prove one of our main results.

Proposition 2. *Let E be a Banach space and let τ be a Hausdorff topology on E . Let $\{T(t) : t \geq 0\}$ be a one-parameter τ -continuous semigroup of mappings on a subset C of E . Let α and β be positive real numbers satisfying $\alpha/\beta \notin \mathbb{Q}$. Then*

$$\bigcap_{t \geq 0} F(T(t)) = F(T(\alpha)) \cap F(T(\beta))$$

holds.

Proof. It is obvious that

$$\bigcap_{t \geq 0} F(T(t)) \subset F(T(\alpha)) \cap F(T(\beta)).$$

We fix $z \in F(T(\alpha)) \cap F(T(\beta))$. Define sequences $\{\alpha_n\}$ in $(0, \infty)$ and $\{k_n\}$ in \mathbb{N} as in Lemma 3. By the assumption, we have

$$T(\alpha_1)z = T(\max\{\alpha, \beta\})z = z \quad \text{and} \quad T(\alpha_2)z = T(\min\{\alpha, \beta\})z = z.$$

If $T(\alpha_j)z = T(\alpha_{j+1})z = z$, then we have

$$T(\alpha_{j+2})z = T(\alpha_{j+2}) \circ T(\alpha_{j+1})^{k_j} z = T(\alpha_{j+2} + k_j \alpha_{j+1})z = T(\alpha_j)z = z.$$

So, by induction, we have $T(\alpha_n)z = z$ for all $n \in \mathbb{N}$. Since $\{\alpha_n\}$ is a positive real sequence converging to 0, we have z is a common fixed point of $\{T(t) : t \geq 0\}$ by Proposition 1. This completes the proof. \square

As a direct consequence of Proposition 2, we obtain the following.

Corollary 1. *Let E be a Banach space and let τ be a Hausdorff topology on E . Let $\{T(t) : t \geq 0\}$ be a one-parameter τ -continuous semigroup of mappings on a subset C of E . Then*

$$\bigcap_{t \geq 0} F(T(t)) = F(T(1)) \cap F(T(\sqrt{2}))$$

holds.

Using Lemma 1, we obtain the following.

Corollary 2. *Let E be a strictly convex Banach space and let τ be a Hausdorff topology on E . Let $\{T(t) : t \geq 0\}$ be a one-parameter τ -continuous semigroup of nonexpansive mappings on a subset C of E . Let α and β be positive real numbers satisfying $\alpha/\beta \notin \mathbb{Q}$, and $F(T(\alpha)) \cap F(T(\beta)) \neq \emptyset$. Then*

$$\bigcap_{t \geq 0} F(T(t)) = \{z \in C : \lambda T(\alpha)z + (1 - \lambda)T(\beta)z = z\}$$

holds for every $\lambda \in (0, 1)$.

Corollary 3. *Let E be a uniformly convex Banach space and let τ be a Hausdorff topology on E . Let $\{T(t) : t \geq 0\}$ be a one-parameter τ -continuous semigroup of nonexpansive mappings on a bounded closed convex subset C of E . Let α and β be positive real numbers satisfying $\alpha/\beta \notin \mathbb{Q}$. Then*

$$\bigcap_{t \geq 0} F(T(t)) = \{z \in C : \lambda T(\alpha)z + (1 - \lambda)T(\beta)z = z\}$$

holds for every $\lambda \in (0, 1)$.

5. CONVERGENCE THEOREMS

Several authors have studied about convergence theorems for one-parameter nonexpansive semigroups; see [1, 3, 11, 16, 18, 20, 22] and others. For example, Suzuki and Takahashi prove in [22] the following: Let C be a compact convex subset of a Banach space E and let $\{T(t) : t \geq 0\}$ be a one-parameter strongly continuous semigroup of nonexpansive mappings on C . Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by

$$x_{n+1} = \frac{\lambda}{t_n} \int_0^{t_n} T(s)x_n ds + (1 - \lambda)x_n$$

for $n \in \mathbb{N}$, where λ is a constant in $(0, 1)$, and $\{t_n\}$ is a sequence in $(0, \infty)$ satisfying

$$\lim_{n \rightarrow \infty} t_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{t_{n+1}}{t_n} = 1.$$

Then $\{x_n\}$ converges strongly to a common fixed point z_0 of $\{T(t) : t \geq 0\}$.

Using Proposition 2, we can prove many convergence theorems to a common fixed point of a one-parameter continuous semigroup of nonexpansive mappings. In this Section, we state some of them. We discuss five types of convergence theorems. Five types are the types of Baillon [2], Krasnoselskii-Mann [14, 15], Ishikawa [12], Browder [6], and Halpern [10]. We first state the following, which are connected with Baillon's type iteration; see pages 63 and 83 in [23].

Theorem 5. *Let E be a Hilbert space and let τ be a Hausdorff topology on E . Let $\{T(t) : t \geq 0\}$ be a τ -continuous semigroup of nonexpansive mappings on a bounded closed convex subset C of E . Fix $\alpha, \beta > 0$ with $\alpha/\beta \notin \mathbb{Q}$. Let $x \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_n = \frac{\sum_{k=1}^n \sum_{\ell=1}^n T(k\alpha + \ell\beta)x}{n^2}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to a common fixed point of $\{T(t) : t \geq 0\}$.

Proof. We note that

$$\sum_{k=1}^n \sum_{\ell=1}^n T(k\alpha + \ell\beta)x = \sum_{k=1}^n \sum_{\ell=1}^n T(\alpha)^k \circ T(\beta)^\ell x.$$

So, $\{x_n\}$ converges weakly to a common fixed point z of $T(\alpha)$ and $T(\beta)$. Such z is a common fixed point of $\{T(t) : t \geq 0\}$ by Proposition 2. This completes the proof. \square

Theorem 6. *Let E be a Hilbert space and let τ be a Hausdorff topology on E . Let $\{T(t) : t \geq 0\}$ be a τ -continuous semigroup of nonexpansive mappings on a bounded closed convex subset C of E . Fix $\alpha, \beta > 0$ with $\alpha/\beta \notin \mathbb{Q}$. Let $x \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_n = \frac{\sum_{k=1}^n \left(\frac{T(\alpha) + T(\beta)}{2} \right)^k x}{n}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to a common fixed point of $\{T(t) : t \geq 0\}$.

Proof. By Theorem 1, $\{x_n\}$ converges weakly to z , which is a fixed point of $(T(\alpha) + T(\beta))/2$. So, by Corollary 3, z is a common fixed point of $\{T(t) : t \geq 0\}$. This completes the proof. \square

We next state the following, which are connected with Krasnoselskii-Mann's type iteration; see Reich [17] and Suzuki [19, 21].

Theorem 7. *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable and let τ be a Hausdorff topology on E . Let $\{T(t) : t \geq 0\}$ be a τ -continuous semigroup of nonexpansive mappings on a bounded closed convex subset C of E . Fix $\alpha, \beta > 0$ with $\alpha/\beta \notin \mathbb{Q}$, and $\kappa, \lambda > 0$ with $\kappa + \lambda < 1$. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and*

$$x_{n+1} = \kappa T(\alpha)x_n + \lambda T(\beta)x_n + (1 - \kappa - \lambda)x_n,$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to a common fixed point of $\{T(t) : t \geq 0\}$.

Proof. By Theorem 2, $\{x_n\}$ converges weakly to z , which is a fixed point of

$$\frac{\kappa}{\kappa + \lambda} T(\alpha) + \frac{\lambda}{\kappa + \lambda} T(\beta).$$

So, by Corollary 3, z is a common fixed point of $\{T(t) : t \geq 0\}$. This completes the proof. \square

Theorem 8. *Let E be a Banach space and let τ be a Hausdorff topology on E . Let $\{T(t) : t \geq 0\}$ be a τ -continuous semigroup of nonexpansive mappings on a compact convex subset C of E . Fix $\alpha, \beta > 0$ with $\alpha/\beta \notin \mathbb{Q}$, and $\lambda \in (0, 1)$. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and*

$$x_{n+1} = \lambda \frac{\sum_{k=1}^n \sum_{\ell=1}^n T(k\alpha + \ell\beta)x_n}{n^2} + (1 - \lambda)x_n,$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \geq 0\}$.

Using Ishikawa's result in [12], we obtain the following.

Theorem 9. *Let E be a Banach space and let τ be a Hausdorff topology on E . Let $\{T(t) : t \geq 0\}$ be a τ -continuous semigroup of nonexpansive mappings on a compact convex subset C of E . Fix $\alpha, \beta > 0$ with $\alpha/\beta \notin \mathbb{Q}$ and $\kappa, \lambda \in (0, 1)$. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and*

$$x_{n+1} = (\lambda T(\alpha) + (1 - \lambda) I) \circ (\kappa T(\beta) + (1 - \kappa) I)^n x_n$$

for $n \in \mathbb{N}$, where I is the identity mapping on C . Then $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \geq 0\}$.

We next state the following, which is connected with Browder's type implicit iteration. We note that

$$x \mapsto (1 - \lambda)Tx + \lambda u$$

is a contractive mapping if T is a nonexpansive mapping and $\lambda \in (0, 1)$. By the Banach contraction principle [4], such mappings have a unique fixed point.

Theorem 10. *Let E be a Hilbert space and let τ be a Hausdorff topology on E . Let $\{T(t) : t \geq 0\}$ be a τ -continuous semigroup of nonexpansive mappings on a bounded closed convex subset C of E . Fix $\alpha, \beta > 0$ with $\alpha/\beta \notin \mathbb{Q}$. Let $u \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_n = \frac{1 - \lambda_n}{2}T(\alpha)x_n + \frac{1 - \lambda_n}{2}T(\beta)x_n + \lambda_n u$$

for $n \in \mathbb{N}$, where $\{\lambda_n\}$ is a sequence in $(0, 1)$ converging to 0. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \geq 0\}$.

We finally state the following, which is connected with Halpern's type explicit iteration; see Wittmann [24].

Theorem 11. *Let E be a Hilbert space and let τ be a Hausdorff topology on E . Let $\{T(t) : t \geq 0\}$ be a τ -continuous semigroup of nonexpansive mappings on a bounded closed convex subset C of E . Fix $\alpha, \beta > 0$ with $\alpha/\beta \notin \mathbb{Q}$. Let $u \in C$ and define a sequence $\{x_n\}$ in C by $x_1 \in C$ and*

$$x_{n+1} = \frac{1 - \lambda_n}{2}T(\alpha)x_n + \frac{1 - \lambda_n}{2}T(\beta)x_n + \lambda_n u$$

for $n \in \mathbb{N}$, where $\{\lambda_n\}$ is a sequence in $[0, 1]$ satisfying the following:

$$\lim_{n \rightarrow \infty} \lambda_n = 0; \quad \sum_{n=1}^{\infty} \lambda_n = \infty; \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$$

Then $\{x_n\}$ converges strongly to a common fixed point of $\{T(t) : t \geq 0\}$.

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